J. Fluid Mech. (1967), *vol.* 27, *part* 2, *pp.* 395–397
Printed in Great Britain

A variational principle for a fluid with a free surface

By J. C. LUKE
California Institute of Technology

(Received 25 February 1966)

The full set of equations of motion for the classical water wave problem in Eulerian co-ordinates is obtained from a Lagrangian function which equals the pressure. This Lagrangian is compared with the more usual expression formed from kinetic minus potential energy.

An expression for the pressure has been used (Clebsch 1859; Hargreaves 1908; Bateman 1944) as the Lagrangian when the equations of motion in an inviscid, incompressible fluid are derived from a variational principle, but it seems not to have been explicitly indicated that a simple extension of this variational principle also provides the boundary conditions appropriate to a free surface. This extension is important in the application to water wave problems, where the non-linear boundary conditions for the free surface are the primary concern. The formulation below is also related to that given by Friedrichs (1933) and Garabedian & Spencer (1952), who used a variational principle to obtain the pressure condition at the free surface in steady flows. The present interest in variational principles for water waves and related problems arose from their use by Whitham (1965, 1966) in the theory of non-linear dispersive waves.

First, for the irrotational case, let $\phi(x,y,t)$ be the velocity potential of a fluid lying between $y = 0$ and $y = h(x,t)$, with gravity acting in the negative y-direction. Then the variational principle is

$$\delta J = \delta \int_{t_1}^{t_2} \int_{x_1}^{x_2} L\, dx\, dt = 0, \quad L = \int_0^{h(x,t)} (\tfrac{1}{2}\phi_x^2 + \tfrac{1}{2}\phi_y^2 + \phi_t + gy)\, dy, \tag{1}$$

where $\phi(x,y,t)$ and $h(x,t)$ are allowed to vary subject to the restrictions $\delta\phi = 0$, $\delta h = 0$ at x_1, x_2, t_1 and t_2. The only change from earlier formulations using an expression for the pressure is that $h(x,t)$ variations are allowed here.

According to the usual procedure in the calculus of variations, (1) becomes

$$\delta J = \int_{t_1}^{t_2}\int_{x_1}^{x_2}\left\{[\tfrac{1}{2}\phi_x^2 + \tfrac{1}{2}\phi_y^2 + \phi_t + gy]_{y=h}\,\delta h + \int_0^{h(x,t)} (\phi_x \delta\phi_x + \phi_y \delta\phi_y + \delta\phi_t)\, dy\right\} dx\, dt = 0. \tag{2}$$

Certain natural boundary conditions arise at $y = h$ and $y = 0$ if the integrated terms are carefully retained when (2) is integrated by parts. Thus

$$\delta J = \int_{t_1}^{t_2}\int_{x_1}^{x_2}\left\{[\tfrac{1}{2}\phi_x^2 + \tfrac{1}{2}\phi_y^2 + \phi_t + gy]_{y=h}\,\delta h + [(-h_x\phi_x + \phi_y - h_t)\,\delta\phi]_{y=h} - [\phi_y\,\delta\phi]_{y=0} - \int_0^h (\phi_{xx} + \phi_{yy})\,\delta\phi\, dy\right\} dx\, dt = 0. \tag{3}$$

First, choose $\delta h = 0$, $[\delta\phi]_{y=0} = [\delta\phi]_{y=h} = 0$; since $\delta\phi$ is arbitrary otherwise, we deduce we have $\phi_{xx}+\phi_{yy} = 0$. Then, since δh, $[\delta\phi]_{y=h}$, $[\delta\phi]_{y=0}$ may be given arbitrary independent values,

$$\tfrac{1}{2}\phi_x^2+\tfrac{1}{2}\phi_y^2+\phi_t+gy = 0 \quad \text{for } y = h, \tag{4}$$

$$-h_x\phi_x+\phi_y-h_t = 0 \quad \text{for } y = h, \tag{5}$$

$$-\phi_{xx}-\phi_{yy} = 0 \quad \text{for } 0 < y < h, \tag{6}$$

$$-\phi_y = 0 \quad \text{for } y = 0. \tag{7}$$

These are the equations for the classical water wave problem.

No satisfactory solution seems known for the general problem of finding suitable Lagrangian functions. For the water wave problem, in particular, the pressure function used in (1) is more productive than the traditional form of the Lagrangian, L^*, equal to kinetic minus potential energy. It is clear that

$$\delta\int_{t_1}^{t_2}\int_{x_1}^{x_2} L^*\,dx\,dt = 0, \quad L^* = \int_0^{h(x,t)} (\tfrac{1}{2}\phi_x^2+\tfrac{1}{2}\phi_y^2-gy)\,dy \tag{8}$$

must give the correct equation within the fluid, for the integrals in (8) and (1) differ by the expression

$$-\int_{t_1}^{t_2}\int_{x_1}^{x_2}\int_0^{h(x,t)} (2gy+\phi_t)\,dy\,dx\,dt, \tag{9}$$

which integrates, leaving only boundary terms. However, (9) contributes boundary terms at $y = h$, so that (8), as it stands, does not give the correct surface conditions.

To see the difference in the boundary conditions, it is necessary instead to relate L^* to the negative of L. From (1) and (8), the integral of $L+L^*$ is

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\int_0^{h(x,t)} (\phi_x^2+\phi_y^2+\phi_t)\,dy\,dx\,dt, \tag{10}$$

which, after integration by parts, becomes

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\Big\{[\phi(-h_x\phi_x+\phi_y-h_t)]_{y=h}-[\phi\phi_y]_{y=0} \\ -\int_0^h \phi(\phi_{xx}+\phi_{yy})\,dy+\frac{\partial}{\partial x}\int_0^h \phi\phi_x\,dy+\frac{\partial}{\partial t}\int_0^h \phi\,dy\Big\}\,dx\,dt. \tag{11}$$

The key to the difference then appears to be conservation of mass. If conservation of mass is introduced by varying ϕ and h only among those functions that satisfy (5)–(7), the difference expression (11) vanishes except for the last two terms, and the last two terms are of no consequence since they contribute only at the x and t boundaries. In this way L^* is made equivalent to L and yields (4), but only at the expense of assuming the other three equations of motion (5)–(7) at the outset.

Only the irrotational case has been treated above. For the rotational case, Clebsch (1859; or see Serrin 1959) expressed the velocity as $\mathbf{u} = \nabla\phi+\alpha\nabla\beta$. Then the variational principle, in a form similar to that given by Bateman (1929; 1944, p. 164), is

$$\delta \iiiint \rho(\phi_t + \alpha\beta_t + \tfrac{1}{2}\mathbf{u}^2 + gy)\, dy\, dx\, dz\, dt = 0. \tag{12}$$

Bateman further generalized the variational principle to barotropic flow, that is, to flows in which the pressure is a function of the density alone. To extend his results to free surfaces it is again merely necessary to include the surface elevation among the quantities to be varied.

The author thanks Prof. G. B. Whitham and Mr G. W. Bluman of the California Institute of Technology for useful suggestions. The author also thanks the National Science Foundation for fellowship support.

REFERENCES

BATEMAN, H. 1929 *Proc. Roy. Soc.* A, **125**, 598.

BATEMAN, H. 1944 *Partial Differential Equations.* Cambridge University Press.

CLEBSCH, A. 1859 *J. reine u. angew. Math.* **56**, 1.

FRIEDRICHS, K. 1933 *Math. Ann.* **109**, 60.

GARABEDIAN, P. R. & SPENCER, D. C. 1952 *J. Ratl. Mech. Anal.* **1**, 359.

HARGREAVES, R. 1908 *Phil. Mag.* (6) **16**, 436.

SERRIN, J. 1959 Mathematical principles of classical fluid mechanics. *Encyclopedia of Physics* (ed. S. Flügge), vol. VIII/1, p. 125. Berlin: Springer-Verlag.

WHITHAM, G. B. 1965 *J. Fluid Mech.* **22**, 273.

WHITHAM, G. B. 1966 *J. Fluid Mech.* **27**, 399.